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A FINITE STRAIN WORK-HARDENING THEORY FOR RATE INDEPENDENT ELASTO-PLASTICITY

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Abstract—A finite strain constitutive theory for a special class of isotropic, rate independent, elastic-plastic materials is proposed. The development proceeds from the assumed existence of a free energy function and the requirement that the postulate of Il'iushin be satisfied for all processes. A precise formulation of the hypothesis of invariance of elastic properties under continued plastic flow, however, leads to an important identity which restricts the class of admissible energy functions and allows for a useful restatement of the known stability requirements. In particular, it is established that a yield function that depends only on the invariants of Kirchhoff stress must necessarily conform to an explicit convexity requirement and act as a potential, in the classical sense, for the plastic strain increments. With this, a complete analogy with the classical small strain development of rate independent elasto-plasticity is established.

INTRODUCTION

Recent considerations in the development of elasto-plastic constitutive theory include attempts to accomodate large deformation through the introduction of finite strain measures and to find some rational justification for the commonly used plastic rate equations. Citing the fundamental paper by Green and Naghdi[1] it is noteworthy that an acceptable structure for the consideration of finite strains has existed only since 1965 while the latter issue is as yet unresolved. Regarding the latter, the initial contribution was made by Drucker[2] in 1951 when he set forth his postulate on plastic stability. But, despite the fact that it seemed to provide a concise and plausible justification for plastic potential theory for the case of infinitesimal strains, its usefulness could not be extended into the finite strain regime. As a result, the need for similar but more generally useful constitutive inequalities, subsidiary to those obtained from standard thermodynamic considerations, was recognized.†

In this regard, considerable effort has been expended in analyzing the implications of the more recent postulate of Il'iushin[4]. This postulate differs from that of Drucker in that it is based on the idea of a closed isothermal strain cycle (rather than a loading cycle) and states simply that the total work done during such a strain cycle should be non-negative, zero if and only if the cycle is completely elastic.

In the present paper the general constitutive structure of Green and Naghdi and the requirement that the postulate of Il'iushin be satisfied for all processes are taken as the starting point for reformulating the specialized finite strain elastic-plastic constitutive theory proposed by Lee[5]. This theory, proposed for the purpose of describing ductile metals during explosive forming processes, is based on the standard model which motivated the classical isotropic work-hardening theory of Hill[6]. Specifically, the ideal material is characterized by the existence of a free energy or elastic strain energy function, the invariance of elastic properties under continued plastic flow, and the incompressibility of the non-recoverable deformation. The latter assumptions make it possible, in essence, to regard plastic flow as a continual updating of the elastic reference configuration. The standard type of yield criteria formulated in terms of an isotropic yield function of the independent state variables is also assumed.

The present formulation differs fundamentally from that of Lee. Most apparent is the reformulation of the kinematics in terms of symmetric positive definite deformation tensors rather than deformation gradients. The initial advantage of this approach is that, by formulating the theory in terms of frame indifferent strain measures, the detailed frame indifference considerations in the original work are made unnecessary. More importantly, by taking the metric tensors associated with an originally orthonormal triad of material elements in the

†A number of these proposed inequalities are critically reviewed by Hill[3].

current and unstressed configurations as the primitive measures of deformation one need not suppose, even locally, the existence of an invertible point transformation defining the unstressed configuration. The existence of such a map would of course predetermine the flatness of the unstressed manifold and thus preclude the existence of local residual stresses. Also, stability requirements such as that of Il'iushin, were not considered in the original work and, as a consequence, the plastic-rate equations were advanced merely as constitutive hypothesis.

The present approach also differs from that of Dafalias[7] in which the restrictions imposed by the Il'iushin postulate on a more general class of elastic–plastic materials are considered. Although Dafalias specifically considers the case for which "the elastic properties are not coupled with plastic deformation" his mathematical formulation of this invariance is, apparently, oversimplified. Here, it is demonstrated that the invariance of elastic properties, in the sense intended by Lee, in fact necessitates, in a Lagrangian energy formulation, a type of "geometric" coupling of elastic and plastic strain tensors quite different from that considered by Dafalias.

The specific requirement that elastic strain energy depend only on the elastic part of the distortion is easily formulated in terms of the invariance of strain energy under arbitrary transformations of the reference configuration. This is seen to require that an allowable energy density be a "coordinate invariant" function of its tensor arguments. An important identity satisfied by such functions then provides the impetus for the completion of the theory. In particular, it is demonstrated that a generalized form of plastic potential theory follows from what is essentially the classical yield criteria.

1. KINEMATICS

The continuing flow or deformation of an ideal isotropic elastic–plastic material from some zero stress reference configuration is described by a smooth one-parameter group of non-singular point transformations

$$x_i = x_i(t, X_j). \tag{1.1}$$

In (1.1) and throughout, the upper and lower case kernel x is used to identify the rectangular Cartesian coordinates of a material particle in the reference and current configurations respectively. In terms of the non-singular deformation gradient

$$x_{i,j} \equiv \frac{\partial x_i}{\partial X_j} \tag{1.2}$$

it is easily seen that material differentials originally proportional to the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ deform continuously into their current configuration $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ according to the rule

$$\mathbf{g}_i = x_{j,i}\mathbf{e}_j. \tag{1.3}$$

The geometry, irrespective of orientation, of the deformed elements is then determined by the standard Green deformation tensor

$$C_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j = x_{k,i}x_{k,j}. \tag{1.4}$$

It is further supposed that the triad of material elements would recover a third configuration $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ with geometry

$$C^p_{ij} \equiv \mathbf{f}_i \cdot \mathbf{f}_j \tag{1.5}$$

upon step removal of stress at that point. The positive definite, symmetric deformation tensor C^p_{ij} is then said to determine, pointwise, the geometry of the so called "unstressed" configuration. In terms of the total and plastic deformation tensors (1.4, 5) the total, plastic and elastic finite strain tensors are then defined as

$$
\begin{aligned}
2E_{ij} &\equiv C_{ij} - \delta_{ij} \\
2E^p_{ij} &\equiv C^p_{ij} - \delta_{ij} \\
2E^e_{ij} &\equiv C_{ij} - C^p_{ij}.
\end{aligned}
\tag{1.6}
$$

After recording the obvious additive property

$$E_{ij} = E^p_{ij} + E^e_{ij} \tag{1.7}$$

their physical significance as they relate the respective lengths ds_0, ds_p and ds of a material element dX_i in the reference, unstressed and current configurations through the quadradic forms

$$
\begin{aligned}
ds^2 - ds_0{}^2 &= 2E_{ij}\, dX_i\, dX_j \\
ds_p{}^2 - ds_0{}^2 &= 2E^p_{ij}\, dX_i\, dX_j \\
ds^2 - ds_p{}^2 &= 2E^e_{ij}\, dX_i\, dX_j
\end{aligned}
\tag{1.8}
$$

is easily verified.

Observe also that it is only necessary to require that the total strain tensor be a compatible strain tensor, i.e. that the Riemann curvature tensor corresponding to C_{ij} vanish pointwise. In general the curvature tensor associated with C^p_{ij} need not vanish and this would in fact provide a test for the existence of local residual stresses.

The sets of directors $\{\mathbf{e}_i\}$, $\{\mathbf{f}_i\}$ and $\{\mathbf{g}_i\}$ also yield to an interpretation relevant to dislocation or slip analysis. In particular, the set of vectors

$$\mathbf{m}_i = (\mathbf{g}_i \cdot \mathbf{f}_j)\mathbf{e}_j; \quad i = 1, 2, 3 \tag{1.9}$$

can be shown to be proportional to the lattice cell edges in the current configuration which, when fully relaxed and stress free, are proportional to the reference traid $\{\mathbf{e}_i\}$. Consequently, when the relaxed and reference traids $\{\mathbf{f}_i\}$ and $\{\mathbf{e}_i\}$ correspond it is seen that

$$\mathbf{m}_i = \mathbf{g}_i \tag{1.10}$$

and hence the crystal deformation is identical to the observed local deformation. In other words, no slip or dislocation has occurred. Thus, the extent to which the relaxed triad $\{\mathbf{f}_i\}$ differs from the reference triad $\{\mathbf{e}_i\}$ may be regarded as a measure of the importance of slip plane activation, relative to pure crystal distortion, in the observed deformation. Clearly, this is reflected in the plastic strain tensor.

The incompressibility of plastic flow is also fundamental to this theory. Thus it is required that the respective volumes of a material element in the unstressed and reference configurations be identical. The fact that this restriction requires that

$$C^p \equiv \det(C^p_{ij}) = 1 \tag{1.11}$$

is a standard result from geometry. The differential form of this restriction,

$$B^p_{ij}\dot{E}^p_{ij} = 0, \quad B^p_{ij} \equiv [C^p_{ij}]^{-1}, \tag{1.12}$$

is obtained by differentiating (1.11) and citing Cramer's rule.

2. CONSTITUTIVE MODEL

The constitutive structure for this theory is fundamentally that of isotropic thermoelasticity with the added complication of a continuously deforming elastic reference configuration. As a consequence of the assumption that this updating of the isotropic elastic reference is the only effect of plastic flow on elastic response, constitutive equations of the form

$$\begin{aligned} \psi &= \hat{\psi}(\mathbf{E}, \mathbf{E}^p, \theta, \nabla_X\theta) \\ s &= \hat{s}(\mathbf{E}, \mathbf{E}^p, \theta, \nabla_X\theta) \\ T_{ij} &= \hat{T}_{ij}(\mathbf{E}, \mathbf{E}^p, \theta, \nabla_X\theta) \\ Q_i &= \hat{Q}_i(\mathbf{E}, \mathbf{E}^p, \theta, \nabla_X\theta), \end{aligned} \tag{2.1}$$

in which θ and $\nabla_X\theta$ represent absolute temperature and referential temperature gradient respectively, are assumed for the specific free energy per unit mass ψ, specific entropy s, symmetric Piola–Kirchoff stress T_{ij}, and the referential heat flux vector Q_j.

The familiar concept of yield is incorporated by assuming the existence of a yield function of the form

$$F = \hat{F}(E_{ij}, \theta, E^p_{ij}, \omega_p), \tag{2.2}$$

in which ω_p represents some scalar hardening parameter reflecting plastic deformation history. Introducing the shorthand

$$G = (E_{ij}, \theta), \quad H = (E^p_{ij}, \omega_p), \tag{2.3}$$

this yield function determines the interface between purely elastic response, characterized by

$$\dot{H} = (\dot{E}^p_{ij}, \dot{\omega}_p) = 0, \tag{2.4}$$

and non-elastic response, according to the following rule: the deformation proceeds elastically whenever

$$F < 0 \tag{2.5}$$

or

$$F = 0 \quad \text{and} \quad \frac{\partial F}{\partial G}\dot{G} < 0, \tag{2.6†}$$

while the deformation proceeds non-elastically with

$$\dot{F} = \frac{\partial F}{\partial G}\dot{G} + \frac{\partial F}{\partial H}\dot{H} = 0 \tag{2.7}$$

whenever

$$F = 0 \quad \text{and} \quad \frac{\partial F}{\partial G}\dot{G} \geq 0. \tag{2.8}$$

Note that, as a consequence of (2.7), the inequality $F \leq 0$ is always satisfied, so that the current value of total strain always lies within or on the projection of the yield surface $F = 0$ in strain space. For this reason it is generally possible to assign a program of purely elastic deformation through any attainable strain state. For those states on the yield surface itself such paths are said to characterize "unloading" processes and have inward pointing tangent vectors at the point of initiation.

It should also be noted that, whereas the exclusion of ω_p as an argument of the constitutive equations (2.1) is physically justified for this material model, the exclusion of temperature gradient as an argument of the yield function is not. Hence, the principle of equipresence is formally violated. However, just as in simple thermoelasticity, it will be seen that only the heat flux equation of those listed in (2.1) can contain explicit dependence on $\nabla_X\theta$. Consequently, it appears that a sufficient justification for this exclusion may be based on an additional physical assumption concerning the insensitivity of yield to heat flux.

†The correspondence: $(\partial F/\partial G)\dot{G} = (\partial\hat{F}/\partial E_{ij})\dot{E}_{ij} + (\partial\hat{F}/\partial\theta)\dot{\theta}$, is intended.

The formal constitutive structure is now completed by assuming that the plastic variables H evolve according to rate laws of the form

$$\left.\begin{aligned}\dot{E}^p_{ij} &= R_{ij}(G, H, \dot{G})\\ \dot{\omega}_p &= r(G, H, \dot{G})\end{aligned}\right\} \dot{H} = R(G, H, \dot{G}), \tag{2.9}$$

and that total strain and absolute temperature can be varied arbitrarily and independently. For definiteness, the constitutive equations (2.1) and the yield function (2.2) are assumed to be continuously differentiable while only piecewise continuity is required of the rate equations (2.9).

The restrictions imposed by the second law of thermodynamics on a constitutive theory of this general type have been considered by Green and Naghdi and Dafalias in the previously cited references. In particular, if the Clausius–Duhem inequality is to be satisfied for all conceivable processes, it is necessary and sufficient to require that free energy be independent of the temperature gradient,

$$\frac{\partial \psi}{\partial \theta_{,k}} = 0, \tag{2.10}$$

that the relationships

$$\begin{aligned} s &= -\frac{\partial \psi}{\partial \theta} \\ T_{ij} &= \rho_0 \frac{\partial \psi}{\partial E_{ij}} \end{aligned} \tag{2.11}$$

hold between the constitutive expressions (2.1), and that the Planck inequality,

$$-\rho_0 \frac{\partial \psi}{\partial E^p_{ij}} \dot{E}^p_{ij} \equiv T^p_{ij} \dot{E}^p_{ij} \geq 0, \tag{2.12}$$

hold for internal dissipation as well as the Fourier inequality,

$$Q_k \theta_{,k} \leq 0, \tag{2.13}$$

for heat flux.

Implicit in (2.12) is the definition for the "thermodynamic tension" T^p_{ij} which in classical small strain theory, where

$$E^e_{ij} = E_{ij} - E^p_{ij}$$

and

$$\psi = \hat{\psi}(E^e_{ij}), \tag{2.14}$$

takes the simple form

$$T^p_{ij} = -\rho_0 \frac{\partial \hat{\psi}}{\partial E^p_{ij}} = \rho_0 \frac{\partial \hat{\psi}}{\partial E_{ij}} = T_{ij}. \tag{2.15}$$

At this point the definition of the scalar hardening parameter ω_p, the counterpart of Lee's "plastic work", is made explicit by equating its rate of change to the non-negative internal dissipation, i.e.

$$\dot{\omega}_p \equiv T^p_{ij} \dot{E}^p_{ij} \geq 0. \tag{2.16}$$

It is easily confirmed that $\dot{\omega}_p$ represents the density for internal entropy production per unit temperature in the absence of temperature gradient or the rate of energy dissipation during plastic flow in the purely mechanical theory.

3. INDEPENDENCE OF ELASTIC PROPERTIES

The independence of elastic properties on plastic strain history is not fully guaranteed by requiring free energy functions to be of the form

$$\psi = \hat{\psi}(\underset{\sim}{E}, \theta, \underset{\sim}{E}^p). \tag{3.1}$$

A more critical restriction on the class of admissible energy functions results from the requirement that the free energy stored in the body depend only on the elastic distortion, i.e. the change in local geometry between the current and unstressed configurations. This is formulated mathematically by requiring the invariance of free energy,

$$\Psi = \int_{V_0} \rho_0 \psi \, \mathrm{d} V_0, \tag{3.2}$$

under an arbitrary change of reference configuration.

In order to assess the implications of this requirement it is convenient to express specific free energy as an explicit function of the deformation tensors (1.4, 5) through the correspondence

$$\psi(\underset{\sim}{C}, \underset{\sim}{C}^p) = \psi\left[\frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I}), \frac{1}{2}(\underset{\sim}{C}^p - \underset{\sim}{I})\right]. \tag{3.3}$$

This is done in order to exploit the fact that the deformation tensors transform according to the covariant rule,

$$\begin{aligned} \bar{C}_{ij} &= \chi^{-1}_{i,m} \chi^{-1}_{j,n} C_{mn} \\ \bar{C}^p_{ij} &= \chi^{-1}_{i,m} \chi^{-1}_{j,n} C^p_{mn}, \end{aligned} \tag{3.4}$$

under an arbitrary transformation of reference configuration,

$$\begin{aligned} \bar{X}_i &= \chi_i(X_j) \\ \chi_{i,j} &\equiv (\partial \chi_i / \partial X_j) \\ J &\equiv \det(\chi_{i,j}) > 0, \end{aligned} \tag{3.5}$$

while the total and plastic strain tensors do not. With this notation, the formal statement of invariance is as follows: the free energy integral (3.2) is locally invariant under arbitrary transformations of reference configuration (3.5) in the sense that

$$\Psi = \int_{V_0} \rho_0 \psi(\underset{\sim}{C}, \underset{\sim}{C}^p) \, \mathrm{d} V_0 = \int_{\bar{V}_0} \bar{\rho}_0 \psi(\bar{\underset{\sim}{C}}, \bar{\underset{\sim}{C}}^p) \, \mathrm{d} \bar{V}_0, \tag{3.6}$$

where the deformation tensors are related through the transformation equations (3.4) and the transformed reference mass density is given by

$$\bar{\rho}_0 = J^{-1} \rho_0. \tag{3.7}$$

By carrying out the simple change of variables

$$\begin{aligned} \int_{\bar{V}_0} \bar{\rho}_0 \psi(\bar{\underset{\sim}{C}}, \bar{\underset{\sim}{C}}^p) \, \mathrm{d} \bar{V}_0 &= \int_{V_0} \bar{\rho}_0 \psi(\bar{\underset{\sim}{C}}, \bar{\underset{\sim}{C}}^p) J \, \mathrm{d} V_0 \\ &= \int_{V_0} \rho_0 \psi(\bar{\underset{\sim}{C}}, \bar{\underset{\sim}{C}}^p) \, \mathrm{d} V_0, \end{aligned} \tag{3.8}$$

it is clear that (3.6) will hold for arbitrary material subdomains if and only if the identity

$$\psi(\bar{C}, \bar{C}^p) = \psi(C, C^p) \tag{3.9}$$

holds pointwise for arbitrary non-singular transformations. In other words, if the free energy is to depend solely on the elastic distortion it is necessary to require that the energy density $\bar{\psi}$ be a "coordinate invariant" function of its tensor arguments. With the selection of an arbitrary one-parameter group of non-singular transformations (3.5) it is a straightforward exercise to show that

$$\frac{d}{d\epsilon}[\psi(\bar{C}, \bar{C}^p)] = -2\left[\frac{\partial\psi}{\partial\bar{C}_{ik}}\bar{C}_{kj} + \frac{\partial\psi}{\partial\bar{C}^p_{ik}}\bar{C}^p_{kj}\right]\chi^{-1}_{m,i}\frac{\partial^2\chi_j}{\partial X_m\partial\epsilon}, \tag{3.10}$$

where ϵ represents the group generator and use has been made of the symmetry of the deformation tensors. As a consequence, it is observed that the coordinate invariance property (3.9) will hold if and only if the identity

$$\frac{\partial\psi}{\partial C_{ik}}C_{kj} + \frac{\partial\psi}{\partial C^p_{ik}}C^p_{kj} = 0 \tag{3.11}$$

is satisfied for all values of the symmetric arguments.

From (3.3) and (1.6) it is also clear that (3.11) is equivalent to the identity

$$\frac{\partial\hat{\psi}}{\partial E_{ik}}C_{kj} + \frac{\partial\hat{\psi}}{\partial E^p_{ik}}C^p_{ij} = 0, \tag{3.12}$$

which may be written as

$$T_{ik}C_{kj} - T^p_{ik}C^p_{kj} = 0$$

in terms of the symmetric Piola–Kirchhoff stress tensor and the thermodynamic tension introduced in (2.12). Multiplication of (3.12) on the right with the inverse plastic deformation tensor, $B^p_{ij} \equiv [C^p_{ij}]^{-1}$, then results in the following useful expression:

$$T^p_{ij} = T_{im}C_{mn}B^p_{nj}. \tag{3.13}$$

With this, the Planck inequality (2.12) takes the alternate form

$$\dot{\omega}_p = \Sigma_{ij}D^p_{ij} \geq 0, \tag{3.14}$$

in terms of new non-symmetric stress and plastic strain rate measures defined by

$$\Sigma_{ij} \equiv T_{ik}C_{kj}; \quad D^p_{ij} \equiv B^p_{ik}\dot{E}^p_{kj}. \tag{3.15}$$

Note in particular that since C_{ij} and B^p_{ij} in fact correspond to the covariant and contravariant metric tensors associated with the Lagrangian coordinate net in the current and unstressed configurations respectively, Σ_{ij} and D^p_{ij} clearly represent the mixed components of true stress and the rate of plastic deformation tensors relative to the appropriate distorted basis.

It is important to note also that the identity (3.12) implies a non-trivial coupling of the total and plastic strain tensors in the argument of a properly invariant energy function. In particular, observe that an energy function of the simple form

$$\psi = \hat{\psi}(E^e) = \hat{\psi}(E - E^p) \tag{3.16}$$

is not properly invariant since, for this type of dependence,

$$T_{ik}C_{kj} - T^p_{ik}C^p_{kj} = 2T_{ik}E^e_{kj}. \quad (3.17)$$

Clearly this may be taken as invariant only to the extent that elastic strains are small.

A particularly simple method for constructing invariant energy functions is to replace the total strain with elastic strain and the Kronecker delta with the inverse plastic deformation tensor in any isotropic elastic energy function. For instance, the quadradic energy function

$$\hat{\psi} = \frac{1}{2}(\lambda\delta_{ij}\delta_{mn} + 2\mu\delta_{im}\delta_{jn})E_{ij}E_{mn}, \quad (3.18)$$

becomes invariant with respect to specification of reference by rewriting it in the form

$$\hat{\psi} = \frac{1}{2}(\lambda B^p_{ij}B^p_{mn} + 2\mu B^p_{im}B^p_{jn})E^e_{ij}E^e_{mn}. \quad (3.19)$$

The identity (3.12) may also be used to restate the restrictions imposed by the stability postulate of Il'iushin which have been investigated by Hill and Rice[8] and, more recently, by Dafalias[7]. In summary, it is shown that a necessary condition for the non-negativity of external work in a closed isothermal strain cycle is that the inequality

$$\frac{\partial}{\partial E^p_{ij}}[\hat{\psi}(\underset{\sim}{E}^1, \underset{\sim}{E}^p) - \hat{\psi}(\underset{\sim}{E}^2, \underset{\sim}{E}^p)]\dot{E}^p_{ij} \geq 0 \quad (3.20)$$

hold for all strains E^2_{ij} on the current projection of the yield surface in strain space, all possible plastic strain rates $\dot{E}^p_{ij}$ at that point, and all other strain states E^1_{ij} attainable through a program of purely elastic deformation. By making use of $(2.12)_2$, $(3.12)_2$, and the defining expressions in (3.15), this inequality may be written in the alternate form

$$[T_{ik}(\underset{\sim}{E}^2, \underset{\sim}{E}^p)C^2_{kj} - T_{ik}(\underset{\sim}{E}^1, \underset{\sim}{E}^p)C^1_{kj}](B^p_{ik}\dot{E}^p_{kj}) = [\Sigma^2_{ij} - \Sigma^1_{ij}]D^p_{ij} \geq 0, \quad (3.21)$$

which is formally identical to that obtained from the Drucker postulate in the small strain development. Notice also that if the elastic region contains $\Sigma_{ij} = 0$ then, for this choice of Σ^1_{ij}, (3.21) reduces the condition of non-negativity of plastic working as required by the second law of thermodynamics.

4. SPECIALIZED YIELD CRITERIA

In view of the stability inequality (3.21) and the analogous developments in small strain theory it is natural to consider an isotropic yield function (2.5) of the specific form

$$F(G, H) = f(\underset{\sim}{\Sigma}, \theta) - c(\omega_p). \quad (4.1)$$

For such a yield function the postulate of Il'iushin is seen to require that the projected yield surface,

$$f(\underset{\sim}{\Sigma}, \theta) = c(\omega_p), \quad (4.2)$$

bound a convex neighborhood of Σ-stress space and that the plastic strain rates project along the outward normal to this yield surface, in the sense that

$$D^p_{ij} = \gamma\frac{\partial f}{\partial \Sigma_{ij}} \quad (\gamma \geq 0), \quad (4.3)$$

whenever yielding occurs. By imposing the additional requirements (3.14) and (1.12) resulting from the second law of thermodynamics and the incompressibility of plastic flow, it is further deduced that the elastic region must contain the origin and, through the expression

$$B^e_{ij}\dot{E}^p_{ij} = D^p_{kk} = \gamma\delta_{ij}\frac{\partial f}{\partial \Sigma_{ij}} = 3\gamma\frac{\partial f}{\partial \Sigma_{kk}} = 0, \tag{4.4}$$

be independent of the isotropic part of Σ-stress. Moreover, the hardening condition (2.7) allows for the explicit determination of the scalar multiplier γ, i.e.

$$\gamma = \begin{cases} \dot{f}\Big/\left(\dfrac{\mathrm{d}c}{\mathrm{d}\omega_p}\Sigma_{ij}\dfrac{\partial f}{\partial \Sigma_{ij}}\right) & f = c \text{ and } \dot{f} \geq 0 \\ 0 & \text{(otherwise)}, \end{cases} \tag{4.5}$$

which is consistently non-negative provided that the work-hardening function c is monotonically increasing. Observe that although (4.3) and (4.5) are formally deduced only for isothermal processes their validity may be extended by hypothesis into the non-isothermal regime without modification. Also note that since

$$\Sigma_{ij} = T_{ik}C_{kj} = x^{-1}_{i,m}x_{n,j}\tau_{mn}, \tag{4.6}$$

where

$$\tau_{ij} = (\rho_0/\rho)\sigma_{ij} \tag{4.7}$$

represents the Kirchhoff stress tensor proportional to Cauchy stress, the invarients of the Σ-stress deviator are identical to those of the stress deviator τ'_{ij}, i.e.

$$\begin{aligned} \mathrm{II} &= \tau'_{ij}\tau'_{ij} = \Sigma'_{ij}\Sigma'_{ij} \\ \mathrm{III} &= \tau'_{km}\tau'_{mn}\tau'_{nk} = \Sigma'_{km}\Sigma'_{mn}\Sigma'_{nk}. \end{aligned} \tag{4.8}$$

Consequently, an isotropic yield function of the form (4.1) differs from a classical yield function only in its dependence on the invariants of Kirchhoff rather than Cauchy stress—a difference which manifests itself exclusively in the presence of elastic dilitation. As a result, it may be easily confirmed that the yield criteria and flow rule based on (4.1) are also insensitive to the specification of reference configuration in the sense that the current evolution of the elastic reference is independent of the level of plastic strain. Thus, it is seen that this theory is sensitive to plastic strain history only through the dependence of the yield and flow rules on accumulated plastic work.

One final point of consistency, since the flow rule (4.3) may be rewritten as

$$\dot{E}^p_{ij} = \gamma C^p_{ik}\frac{\partial f}{\partial \Sigma_{kj}}, \tag{4.9}$$

it must necessarily follow that

$$C^p_{ik}\frac{\partial f}{\partial \Sigma_{kj}} = C^p_{jk}\frac{\partial f}{\partial \Sigma_{ki}}. \tag{4.10}$$

This may be confirmed in light of the fundamental identity (3.12) and the specific dependence of f on the stress invariants.

5. CONCLUSION

The essential feature of this theory is the fact that the current state is completely determined by the absolute temperature, accumulated plastic work and the instantaneous elastic distortion. Dependence on the plastic distortion is eliminated, as required in the theory proposed by Lee, by demanding the invariance of constitutive equations under transformations of reference. This imposes an important restriction on the class of admissible free energy functions. A restatement of a restriction imposed by the Il'iushin stability postulate then leads to a generalized form of plastic potential theory for a specific class of yield functions precisely as in the classical development. This completes the formulation of a simple, Il'iushin stable, finite strain, elastic–plastic constitutive theory incorporating all of the qualitative features of the model proposed by Lee. Implementation of this theory depends only on specification of a coordinate invariant free energy function ψ, a yield function f depending on the absolute temperature and the invariants of the Kirchhoff stress deviator, and a monotone increasing work-hardening function c.

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